

Anisotropic fluids in the case of stationary and axisymmetric spaces of general relativity

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1 Introduction

In the context of general relativity the solutions describing a rotating fluid body are of great importance. They permit to construct realistic models of stars and compare the theoretical predictions to actual observational data concerning rotating stars. Thus the existence of rotating fluid metrics is the first step for the realization of such a project, the second being the matching of the interior solution to an asymptotically flat vacuum metric at a boundary surface of zero pressure.

Currently there is no global spacetime model available for a rotating fluid body in an asymptotically flat vacuum. The problem arises because very few

solutions representing a rotating fluid are known, the most important being the Wahlquist metric [1],[2], which has been studied for three decades, but it has not been succeeded to match this metric to some asymptotically flat vacuum exterior or to prove the existence of such an exterior metric rigourously. Recently Bradley et al [3], have proven the impossibility of matching the Wahlquist metric to a vacuum exterior in the slow rotation limit.

I think that such proofs do not tackle the problem for many reasons: (1) the choice of the exterior vacuum is not unique, (2) a very interesting aspect of the Wahlquist metric is that within the range of a parameter the surface of zero pressure is prolate along the axis of rotation and this indicate the action of an external force [4], then obviously it will be more meaningful to match the Wahlquist metric to some other matter metric (that of a null fluid) and not directly to a vacuum metric.

Also we have to notice that the equation of state for the Wahlquist metric is $e + 3p = \text{const}$ and this equation has to be interpreted, otherwise the physical content of the solution will remain obscure. The possible directions of research are the following: (1) try to match the Wahlquist metric to that of a null fluid and this later to a vacuum exterior, giving in the same time an interpretation for the equation of state $e + 3p = \text{constant}$ (2) find new solutions describing rigidly rotating or differentially rotating [5],[6] perfect fluid and match them to appropriate vacuum exterior (3) drop the assumption of perfect fluid and assume anisotropy which is suggested by the complexity of the strong interactions in certain density ranges [7]. The anisotropy of fluids in the context of General Relativity is already used in the domain of exact solutions [8],[9]. Also we have to notice that Florides [10] used the Florides-Synge method [11] to show that up to κ^5 (where κ is some small parameter) the Kerr metric may be matched to an interior solution describing a rotating body of non perfect fluid (with anisotropic pressures) and in another paper Florides [12] used the same method to match the Kerr metric up to κ^5 to a rigidly rotating oblate spheroid with anisotropic pressures.

These results gave us the idea to study the anisotropic fluids in the case of axisymmetric stationary spaces of General Relativity and because of the complexity of Einstein's equations in this case we begin by considering a more "special case" the Carter's family of metrics. In section 2 we present a brief review of the Newman-Penrose (NP) formalism in the complex vectorial rep-

resentation of Cahen, Debever, Defrise [13],[14],[15] in which we perform our calculation and we present Carter's family of metrics. In section 3 we write the energy-momentum tensor of an anisotropic fluid and we calculate the components of the Ricci traceless tensor and the Weyl tensor, we try then to solve the resulting equations by imposing some kind of equation of state . In section 4 we study the solution obtained and we give some perspectives concerning the continuation of our work.

2 NP formalism and Carter's family of metrics

The Carter's family of metrics [16],[17] can be characterised by the existence of a second rank Killing tensor with two double eigenvalues λ_1, λ_2 and a two parameter Abelian isometry group G_2 , with nonnull surfaces of transitivity with orbits which can be time-like or space-like. We have studied the full family of Carter's metric in the presence of a perfect fluid [18] and the member $[\tilde{A}]$ of the family (λ_1, λ_2 are not constant) in [19] where we obtain a generalization of the Wahlquist solution. In the reference someone can find a short resumé of the relation between Carter's spaces and the Hauser-Mahlot spaces and the implications of the existence of a second rank Killing tensor on the separability of the Hamilton-Jacobi equation for the geodesics. This Killing tensor which characterises Carter' family $[\tilde{A}]$ of solutions can be written in a local coordinate system:

$$K_{ij} = \lambda_1(n_i l_j + n_j l_i) + \lambda_2(\bar{m}_i m_j + m_j \bar{m}_i) \quad (1)$$

where the covariant vectors l_i and n_i are real and null while the null vectors m_i, \bar{m}_i are complex conjugate. the functions λ_1, λ_2 are the two double eigenvalues of the Killing tensor and they are real. The four vectors form a covariant null tetrad and the metric can be put in the form

$$ds^2 = 2(\theta^1 \theta^2 - \theta^3 \theta^4) \quad (2)$$

where

$$\theta^1 = n_i dx^i, \theta^2 = l_i dx^i, \theta^3 = -\bar{m}_i dx^i, \theta^4 = -m_i dx^i \quad (3)$$

A basis for the space of complex self-dual 2-forms is given by

$$Z^1 = \theta^1 \wedge \theta^2, Z^2 = \theta^1 \wedge \theta^3 - \theta^2 \wedge \theta^4, Z^3 = \theta^4 \wedge \theta^2 \quad (4)$$

The components of the metric in this base are

$$\gamma^{\alpha\beta} = 4(\delta_{(1}^{\alpha}\delta_{3)}^{\beta} - \delta_2^{\alpha}\delta_2^{\beta}) \quad (5)$$

The complex connection 1-forms are defined by

$$dZ^{\alpha} = \sigma_{\beta}^{\alpha} \wedge Z^{\beta} \quad (6)$$

Greek indices=1,2,3, Latin indices=1,2,3,4 and the vectorial connection 1-form is defined by

$$\sigma_{\alpha} = \frac{1}{8} e_{\alpha\beta\gamma} \gamma^{\gamma\delta} \sigma_{\delta}^{\beta} \quad (7)$$

where $e_{\alpha\beta\gamma}$ is the three dimensional permutation symbol. The tetrad components $\sigma_{\alpha} = \sigma_{\alpha a} \theta^a$ are 12 complex valued functions which are exactly the NP spin coefficients:

$$\sigma_{\alpha a} = \begin{pmatrix} \kappa & \tau & \sigma & \rho \\ \epsilon & \gamma & \beta & \alpha \\ \pi & \nu & \mu & \lambda \end{pmatrix} \quad (8)$$

The complex curvature 2-forms Σ_{β}^{α} are defined by

$$d\sigma_{\beta}^{\alpha} - \sigma_{\gamma}^{\alpha} \wedge \sigma_{\delta}^{\gamma} = \Sigma_{\delta}^{\beta} \quad (9)$$

and the vectorial curvature 2-form by

$$\Sigma_{\alpha} = \frac{1}{8} e_{\alpha\beta\gamma} \gamma^{\gamma\delta} \Sigma_{\delta}^{\beta} \quad (10)$$

On expanding Σ_{α} in the basis $[Z^{\alpha}, \bar{Z}^{\alpha}]$ one obtains:

$$\Sigma_{\alpha} = (C_{\alpha\beta} - \frac{1}{6} R \gamma_{\alpha\beta}) Z^{\beta} + E_{\alpha\bar{\beta}} Z^{\bar{\beta}} \quad (11)$$

where the quantities $C_{\alpha\beta}$ and $E_{\alpha\bar{\beta}}$ are related to the NP curvature components Ψ_A and Φ_{AB} as follows:

$$C_{\alpha\beta} = \begin{pmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{pmatrix}, \quad E_{\alpha\bar{\beta}} = \begin{pmatrix} \Phi_{00} & \Phi_{01} & \Phi_{02} \\ \Phi_{10} & \Phi_{11} & \Phi_{12} \\ \Phi_{20} & \Phi_{21} & \Phi_{22} \end{pmatrix} \quad (12)$$

The Carter's family $[\bar{A}]$ of metrics can be written as follows

$$ds^2 = (\Phi + \Psi) \left\{ \frac{fE^2}{(B-A)^2} (dt + Adz)^2 - \frac{H^2}{(B-A)^2} (dt + Bdz)^2 - \frac{f(\Psi_y dy)^2}{4G^2} - \frac{(\Phi_x dx)^2}{4F^2} \right\} \quad (13)$$

where

$$\lambda = \Phi(x), \lambda_2 = \Psi(y), \quad \Phi_x = \frac{d\Phi}{dx} \quad \Psi_y = \frac{d\Psi}{dy} \quad (14)$$

$$A = A(x), \quad H = H(x) \quad F = F(x), \quad (15)$$

$$B = B(y), \quad E = E(y) \quad G = G(y) \quad (16)$$

$f = +1$ there is one time-like Killing vector $\frac{\partial}{\partial t}$ and one space like Killing vector $\frac{\partial}{\partial z}$, this is the axisymmetric case.

$f = -1$ the Killing vectors are both space-like.

We consider in this paper only the $f = +1$ case. For all plausible energy-momentum tensors the traceless Ricci components have to be real, the only complex components is Φ_{01} and its imaginary part is equal to

$$\frac{3}{4} \frac{i}{(\Phi + \Psi)} \frac{4GF}{\Phi_x \Psi_y} \left\{ \ln \left(\frac{\Phi + \Psi}{B - A} \right) \right\}_{xy} \quad (17)$$

where $\{ \}_{xy} = \frac{\partial^2}{\partial x \partial y} \{ \}$. The vanishing of this expression is the necessary and sufficient condition for the separability of the Schrödinger equation:

$$\left\{ \ln \left(\frac{\Phi + \Psi}{B - A} \right) \right\}_{xy} = 0 \quad (18)$$

The only solution of (16) that has been used in the literature is the most obvious one

$$B(y) = \Psi(y), A(x) = -\Phi(x) \quad (19)$$

The general solution of (16) is given by

$$A(x) = \frac{l_i \Phi + l_2}{l_3 \Phi + l_4}, B(y) = \frac{l_1 \Psi - l_2}{l_3 \Psi - l_4} \quad (20)$$

In [19] we claimed that solution (18) could lead in the presence of a perfect fluid, to different metrics than that of Wahlquist or the generalization of Wahlquist obtained there. Unfortunately the calculations proved that there is no new solution in the case of a perfect fluid energy momentum tensor. So we present this generalization of the Wahlquist solution with some minor changes in the constants, which clarify the reduction to the Wahlquist solution, we present also the vacuum metric of the Carter's family given by (13) and how we get the Kerr metric from (13) in order to compare these metrics with the obtained for the anisotropic fluids.

We generalize the Wahlquist solution as follows

$$ds^2 = \frac{1}{\zeta^2 + \xi^2} \{ E^2(\zeta) [dt - (l\xi^2 - \frac{k_2 - l}{2q^2})dz]^2 - H^2(\xi) [dt + (l\zeta^2 - \frac{k_2 - l}{2q^2})dz]^2 \} - (\zeta^2 + \xi^2) \left[\frac{d\zeta^2}{E^2(\zeta)(1 - q^2\zeta^2)} + \frac{d\xi^2}{H^2(\xi)(1 + q^2\xi^2)} \right]$$

$$E^2(\zeta) = -\frac{a^2}{q^2} \zeta(1 - q^2\zeta^2)^{\frac{1}{2}} \sin^{-1}(q\zeta) + (B + 2\Gamma q^2)\zeta^2 + p_1\zeta(1 - q^2\zeta^2)^{\frac{1}{2}} - \Gamma, \quad (21)$$

$$H^2(\xi) = \frac{a^2}{q^2} \xi(1 - q^2\xi^2)^{\frac{1}{2}} \sinh^{-1}(q\xi) - (B + 2\Gamma q^2)\xi^2 + p_2\xi(1 + q^2\xi^2)^{\frac{1}{2}} - \Gamma \quad (22)$$

and

$$B = \frac{1}{\beta} (a^2 + \frac{\gamma_1 + 1}{\gamma_2 r_0^2}) \quad (23)$$

$$\Gamma = \frac{1}{\beta^2(\beta + 1)\gamma_2 r_0^2} [a^2(1 - \beta)\gamma r_0^2 + 2(\gamma_1 - \beta)] \quad (24)$$

$$k_2 = \gamma_2 r_0^2, l = \gamma_2 \beta r_0^2 \quad (25)$$

$$\beta = (1 - 4q^2)^{\frac{1}{2}} \quad (26)$$

and $\gamma_1 = \gamma_1(a, q), \gamma_2 = \gamma_2(a, q)$

$$\lim_{a, q \rightarrow 0} \gamma_1 = 0, \lim_{a, q \rightarrow 0} \gamma_2 = 1 \quad (27)$$

The constants of this metric are $a, q, r_0, p_1, p_2, \gamma_1, \gamma_2$. Obviously γ_1, γ_2 are not independent constants, they depend on a, q which are the fluid constants: if $a = q = 0$ the fluid disappears, r_0 is the radius of the system of oblate spheroidal coordinates and if $p_1 = p_2 = 0$ there is no ring singularity.

We get the Wahlquist solution setting

$$\gamma_2 = \frac{2(1 - \beta)}{\beta(\beta + 3) - a^2 r_0^2(1 - \beta)}, \gamma_1 = \frac{(1 - \beta)[\beta^2(\beta - 2) - a^2 r_0^2(1 + 2\beta)]}{\beta(\beta + 3) - a^2 r_0^2(1 - \beta)} \quad (28)$$

$$\xi_A^2 = \frac{2}{\beta(\beta + 1)}, \delta = \pm \beta \gamma_2 r_0^2, z = \tilde{z} r_0, a^2 r_0^2 = \frac{1}{k^2} \quad (29)$$

ξ_A, δ are the constants used by Wahlquist, they are not independent, but Wahlquist did not show explicitly their dependence in his paper.

The relation of the Wahlquist coordinates ζ, ξ with x, y of the metric (13) is

$$x^2 = l\xi^2 - \frac{k_2 - l}{2q^2}, \quad y^2 = l\zeta^2 + \frac{k_2 - l}{2q^2} \quad (30)$$

The vacuum solution for the metric (13) is given by:

$$B(y) = \Psi(y) = y^2, G^2 = y^2 E^2(y), A(x) = -\Phi(x) = -x^2, F^2 = x^2 H^2(x) \quad (31)$$

$$E^2(y) = \frac{1}{2}ay^2 + by + c, H^2(x) = -\frac{1}{2}ax^2 + dx + c \quad (32)$$

The Kerr metric can be obtained from (28) if we set:

$$a = 2, d = 0, b = -2m, c = \alpha^2 y = r, x = \alpha \cos \theta \quad (33)$$

r, θ are the Boyer, Lindquist coordinates then we have that

$$E^2 = r^2 - 2mr + \alpha^2, H^2 = \alpha^2 \sin^2 \theta \quad (34)$$

$$x^2 + y^2 = r^2 + \alpha^2 \cos^2 \theta \quad (35)$$

and the definition of new coordinates \hat{t}, ϕ :

$$dz = ad\phi, \quad dt = d\hat{t} + d\phi \quad (36)$$

brings the metric to its final form.

3 Carter's metric $[\tilde{A}]$ in the presence of an anisotropic fluid

We assume that the energy momentum tensor is locally anisotropic and in the tangent space can be put in the form:

$$T_j^i = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p_y & 0 & 0 \\ 0 & 0 & -p_z & 0 \\ 0 & 0 & 0 & -p_x \end{pmatrix} \quad (37)$$

This is most general case for a second rank tensor, it appears to have four distinct eigenvalues (one positive and three negative), if we use the tetrad defined by (3), we can write the energy momentum tensor as follows:

$$\begin{aligned}
T_{ij} = & \frac{1}{2}(e + p_y)(n_i n_j + l_i l_j) + \frac{1}{2}(p_z - p_x)(\bar{m}_i \bar{m}_j + m_i m_j) + \\
& + \frac{1}{4}(e - p_y + p_z + p_x)(n_i l_j + l_i n_j + \bar{m}_i m_j + m_i \bar{m}_j) + \\
& + \frac{1}{4}(e - p_y - p_z - p_x)(n_i l_j + l_i n_j - \bar{m}_i m_j - m_i \bar{m}_j)
\end{aligned} \tag{38}$$

The Einstein equations

$$R_{ij} - \frac{1}{2}Rg_{ij} = T_{ij} \tag{39}$$

give the following expressions for the components of the traceless Ricci tensor in the NP notation:

$$\Phi_{01} = \Phi_{12} = 0 \tag{40}$$

$$\Phi_{00} = \frac{1}{4}(e + p_y), \quad \Phi_{02} = \frac{1}{4}(p_z - p_x) \tag{41}$$

$$2\Phi_{11} = \frac{1}{4}(e - p_y + p_z + p_x) \quad , \quad 6\Lambda = \frac{1}{4}(e - p_y - p_z - p_x) \tag{42}$$

The Carter metric $[\tilde{A}]$ can be written in the following way:

$$\begin{aligned}
ds^2 = & \frac{1}{x^2 + y^2} \{ E^2(y)[dt - x^2 dz]^2 - H^2(x)[dt + y^2 dz]^2 \} - \\
& - (x^2 + y^2) \left[\frac{y^2 dy^2}{G^2} + \frac{x^2 dx^2}{F^2} \right]
\end{aligned} \tag{43}$$

The components of the traceless Ricci tensor and the Weyl tensor in the NP notation for the metric (36) are:

$$\Phi_{01} = \Phi_{21} = \frac{HE}{4(x^2 + y^2)^3} \left[\frac{(T^2)_y}{2y} - \frac{(\Pi^2)_x}{2x} \right], \tag{44}$$

$$\Phi_{00} = \Phi_{22} = \frac{E^2}{2(x^2 + y^2)^3} \left[\frac{(T^2 + \Pi^2)}{x^2 + y^2} - \frac{(T^2)_y}{2y} \right] \tag{45}$$

$$\Phi_{02} = \Phi_{20} = \frac{H^2}{2(x^2 + y^2)^3} \left[\frac{(T^2 + \Pi^2)}{x^2 + y^2} - \frac{(\Pi^2)_x}{2x} \right] \tag{46}$$

$$\Psi_1 = \Psi_3 = \frac{HE}{2(x^2 + y^2)^3} \left[-\frac{2(T^2 + \Pi^2)}{x^2 + y^2} + \frac{(T^2)_y}{2y} + \frac{(\Pi^2)_x}{2x} \right] \tag{47}$$

$$\Psi_0 = \Psi_4 = 0 \tag{48}$$

Φ_{11} , 6Λ , Ψ_2 are lengthy and will not be given here, we have used the notations:

$$T(y) = \frac{G(y)}{E(y)}, \quad \pi(x) = \frac{F(x)}{H(x)}, \quad (T^2)_y = \frac{dT^2}{dy} \dots \tag{49}$$

If we impose conditions (35) to the expressions (37)-(41) we get that:

$$\frac{(T^2)_y}{2y} - \frac{(\Pi^2)_x}{2x} = 0 \quad (50)$$

The solution of this equation is

$$T^2(y) = \frac{G^2(y)}{E^2(y)} = k_2 y^2 + k_0, \Pi^2(x) = \frac{F^2(x)}{H^2(x)} = k_2 x^2 + l_0 \quad (51)$$

Substitution of (44) in (38)-(40) implies that:

$$\Phi_{00} = \Phi_{22} = \frac{E^2(y)(k_0 + l_0)}{2(x^2 + y^2)} \quad (52)$$

$$\Phi_{02} = \Phi_{20} = \frac{H^2(x)(k_0 + l_0)}{2(x^2 + y^2)} \quad (53)$$

$$\Psi_1 = \Psi_3 = -\frac{H(x)E(y)(k_0 + l_0)}{2(x^2 + y^2)} \quad (54)$$

$2\Phi_{11}, 6\Lambda$ are given in appendix. Expression (47) permit the following classification

$$\Psi_1 = \Psi_3 = 0 \quad \text{or} \quad k_0 + l_0 = 0, \Psi_2 \neq 0 \quad (55)$$

$$\text{and} \quad (56)$$

$$\Psi_1 = \Psi_3 \neq 0, \quad k_0 + l_0 \neq 0 \quad \Psi_2 \neq 0 \quad (57)$$

Relations (48) imply that the metric is of type D in the Petrov classification and it is determined by 928), this metric is the vacuum case of Carter's family $[\tilde{A}]$. Relations (49) imply that the metric is of type I in the Petrov classification provided that $9\Psi_2^2 \neq 16\Psi_1^2$, this metric has been obtained by imposing the anisotropic fluid energy momentum tensor (33). It is obvious that in our approach we have still two unknown functions $E^2(y), H^2(x)$ and no other equation available to define them. We have to impose a supplementary condition and this condition could be an equation of state between the pressure and the rest energy e , but we have to pay attention to the fact that the quantities p_x, p_y, p_z contain contribution from fluid pressure as well as other stresses and that generally these quantities depend on additional variables (such as entropy magnetic fields etc). The supplementary condition that we impose is justified only by the fact that we can write down an equation which can be solved and define in this way $E^2(x), H^2(x)$:

$$e + p_z = 2(p_x - p_y) \quad (58)$$

Using (50) , (35) and the expressions (45),(46) and those of the appendix of the appendix we can write the following equation:

$$(y^2 + x^2)(k_2 y^2 + k_0) \frac{(E^2)_{yy}}{y^2} - (4k_2 y^4 + 5k_0 y^2 + k_0 x^2) \frac{(E^2)_y}{y^3} + 4k_2 E^2 - (y^2 + x^2)(k_2 x^2 + l_0) \frac{(H^2)_{xx}}{x^2} - (4k_2 x^4 + 5k_0 x^2 + l_0 y^2) \frac{(H^2)_x}{x^3} + 4k_2 H^2 = 0 \quad (59)$$

If we differentiate twice (51) with respect to x and twice with respect to y we get the separation of x and y

$$(k_2 y^2 + k_0) \frac{(E^2)_{yy}}{y^2} - k_0 \frac{(E^2)_y}{y^3} = -\frac{d}{2n} y^2 + a_1 y + a_0, \quad (60)$$

$$(k_2 x^2 + l_0) \frac{(H^2)_{xx}}{x^2} - l_0 \frac{(H^2)_x}{x^3} = -\frac{d}{2n} x^2 + a_1 x + a_0 \quad (61)$$

the constant d is the constant of separation and the constants a_1, a_0, b_1, b_0, n are constants of integration. the integration by parts of (52),(53) result two linear differential equations of first order:

$$(k_2 y^2 + k_0)(E^2)_y - k_2 y E^2 = -\frac{d}{8n} y^5 + \frac{1}{3} a_1 y^4 + \frac{1}{2} a_0 y^3 + r_1 y \quad (62)$$

$$(k_2 x^2 + l_0)(H^2)_x - k_2 x H^2 = -\frac{d}{8n} x^5 + \frac{1}{3} b_1 x^4 + \frac{1}{2} b_0 x^3 + r_2 x \quad (63)$$

These equations can be easily integrated and the expressions for E^2, H^2 are substituted in (51) which has to be satisfied identically. Finally we get:

$$E^2 = q_1 \sqrt{k_2 y^2 + k_0} - \frac{d}{24k_2 \eta} y^4 + \frac{1}{2k_2} \left(\frac{k_0 d}{3k_2 \eta} + a_0 \right) y^2 - \frac{r_1}{k_2} + \frac{k_0^2 d}{3k_2^3 \eta} + \frac{a_0 k_0}{k_2^2} \quad (64)$$

$$H^2 = q_2 \sqrt{k_2 x^2 + l_0} - \frac{d}{24k_2 \eta} x^4 + \frac{1}{2k_2} \left(\frac{l_0 d}{3k_2 \eta} - a_0 \right) x^2 - \frac{r_1}{k_2} + \frac{l_0^2 d}{3k_2^3 \eta} - \frac{a_0 l_0}{k_2^2} \quad (65)$$

where $q_1, q_2, d, k_2, \eta, r_1, r_2, a_0, k_0, l_0$ are arbitrary constants.

4 Properties of the solution

Our approach of anisotropic fluids is based on the form of the energy-momentum tensor (38). This form has the following disadvantages: there

is no natural way to define the hydrostatic pressure and consequently it is not possible to obtain the zero pressure surface which is necessary to construct a realistic star model, also we have no four-velocity for a comoving with the fluid observer so we are not able to characterize the motion of the fluid.

On the other hand this solution (metric (43) with (51) and (64),(65)) it is the first exact solution which can describe an anisotropic fluid. Wahlquist in [2] mentioned the possibility for such solutions but he has not succeeded to solve the differential equations which are satisfied by the two remaining unknown functions. Finally this solution reduces to the vacuum family $[\tilde{A}]$ of Carter's spaces if we impose the condition:

$$k_0 + l_0 = 0 \quad (66)$$

Some evidence that our solution describes a realistic anisotropic fluid comes from the fact that it is compatible with the existence of a rotation axis which satisfies the condition of elementary flatness. Also our solution can in principle satisfy the strong energy conditions:

The position of the rotation axis is given by the vanishing of the axial Killing vector:

$$x = 0 \quad \text{and} \quad H^2(x = 0) = 0 \quad (67)$$

these relations imply that

$$q_2 \sqrt{l_0} - \frac{r_1}{k_2} + \frac{l_0^2 d}{3k_2^3} - \frac{a_0 l_0}{k_2^2} = 0 \quad (68)$$

and that

$$l_0 > 0 \quad (69)$$

The elementary flatness of the rotation axis is guaranteed by the regularity condition:

$$\frac{X_{,i} X^{,i}}{4X} \rightarrow 1, \quad \text{on the axis} \quad (70)$$

where

$$X = U_i U^i, \quad U = \frac{\partial}{\partial z} \quad (71)$$

It is remarkable that relation (60) ensures (62) also!!

If finally we consider expressions for the eigenvalues of the energy- momentum tensor (given in appendix) we can prove the following statement. If we suppose that:

$$e + p_z = g_1^2, g_1 \in \mathbf{R} \quad (72)$$

$$e + p_y = g_2^2, g_2 \in \mathbf{R} \quad (73)$$

then we can show that:

$$e + p_x = g_2^2 + \frac{1}{2}g_1^2 \quad (74)$$

$$e + p_x + p_y + p_z = g_1^2 - \frac{1}{4}d \quad (75)$$

The positivity of the expressions $e + p_z, e + p_y, e + p_x$ and $e + p_x + p_y + p_z$ is nothing else but the strong energy conditions!! Then we have proved that the satisfaction of the two of the strong energy conditions implies the validity of the remaining two conditions (for $d < 0$ this true also for all possible values of g_1).

Obviously we have solved a part of the problem of finding a global space time model for a rotating fluid. we have to define now in a consistent way a surface of zero pressure and then match our solution of anisotropic fluid to the vacuum spaces of Carter's family $[\tilde{A}]$. The vacuum spaces $[\tilde{A}]$ of Carter have not been studied until now despite the fact that they are an important generalization of Kerr metric and this necessity is stipulated by Carter in [16]. We think that the existence of four arbitrary constants in Carter's spaces $[\tilde{A}]$ makes them more appropriate for the matching with an interior metric. Finally the determination of the zero pressure surface will permit us to answer if there is a singularity (the $x = y = 0$ singularity) inside the physical region of the fluid.

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5 APPENDIX

$$\begin{aligned}
2\Phi_{11} = & \frac{(k_2 y^2 + k_0) \frac{d^2 E^2}{dy^2}}{4y^2(x^2 + y^2)} - \frac{(4k_2 y^4 + 5k_0 y^2 + k_0 x^2) \frac{dE^2}{dy}}{4y^3(x^2 + y^2)^2} + \frac{(2k_2(x^2 + y^2) + 3(l_0 + k_0))E^2(y)}{2(x^2 + y^2)^3} + \\
& - \frac{(k_2 x^2 + l_0) \frac{d^2 H^2}{dx^2}}{4x^2(x^2 + y^2)} + \frac{(l_0 y^2 + 4k_2 x^4 + 5l_0 x^2) \frac{dH^2}{dx}}{4x^3(x^2 + y^2)^2} - \frac{(2k_2(x^2 + y^2) + 3(l_0 + k_0))H^2(x)}{2(x^2 + y^2)^3} \quad (76)
\end{aligned}$$

$$\begin{aligned}
-6\Lambda = & \frac{(k_2 y^2 + k_0) \frac{d^2 E^2}{dy^2}}{4y^2(x^2 + y^2)} - \frac{k_0 \frac{dE}{dy}}{4y^3(x^2 + y^2)} + \frac{(l_0 + k_0)E^2(y)}{2(x^2 + y^2)^3} + \\
& \frac{(k_2 x^2 + l_0) \frac{d^2 H^2}{dx^2}}{4x^2(x^2 + y^2)} - \frac{l_0 \frac{dH}{dx}}{4x^3(x^2 + y^2)} - \frac{(k_0 + l_0)H^2(x)}{2x^3(x^2 + y^2)^3} \quad (77)
\end{aligned}$$